Finite-temperature effects in a Robertson-Walker universe

1978 J. Phys. A: Math. Gen. 11 L77

(http://iopscience.iop.org/0305-4470/11/4/003)

J. Phys. A: Math. Gen., Vol. 11, No. 4, 1978. Printed in Great Britain.

LETTER TO THE EDITOR

Finite-temperature effects in a Roberston–Walker universe

Gerard Kennedy

Department of Theoretical Physics, The University of Manchester, Manchester M13 9PL, UK

Received 20 February 1978

Abstract. The concept of thermal equilibrium in a Robertson–Walker universe is discussed. A finite-temperature Green function is constructed and used to calculate $\langle\hat{T}_{00}\rangle_{\beta_0}$. A recognised static form results.

1. Introduction

Although our usual concepts of thermal equilibrium require us to restrict the space–time, on which a thermal gas propagates, to be static (or, more precisely, stationary) (Dowker and Critchley 1977, Gibbons and Perry 1977, Dowker and Kennedy 1978), Israel (1972) has found equilibrium distribution functions for massless particles when the space–time is merely conformally stationary. It has been mentioned by Gibbons and Perry (1977) that such an approach can be used in particular to define finite-temperature field theory on Robertson–Walker (RW) metrics. In the present Letter we shall illustrate this conformal method by calculating $\langle\hat{T}_{00}\rangle_{\beta_0}$ for a closed RW metric and try to indicate that this is not merely a technical exercise but does have physical significance.

2. Finite-temperature Green function

We start with the expression for the line element of the RW universe in the usual (co-moving) frame

$$ds^2 = dt^2 - R^2(t)\, d\sigma^2 \tag{1}$$

where $R(t)$ is the usual radius function and $d\sigma^2$ is the line element on the spatial S^3. In its form (1) the metric is obviously not static but, as is well known (Hawking and Ellis 1973), it is conformally related to the metric of the (static) Einstein universe,

$$ds^2 = \left(\frac{R(t)}{a}\right)^2 (d\tau^2 - a^2\, d\sigma^2) \tag{2}$$

where the Einstein time, τ, is related to the RW time, t, by

$$\frac{dt}{d\tau} = \frac{R(t)}{a}. \tag{3}$$

For simplicity we consider a massless conformally invariant scalar field propagating on the RW metric. This field satisfies

$$(\Box+\tfrac{1}{6}R)\phi(x)=0, \tag{4}$$

and the corresponding Green function satisfies

$$(\Box+\tfrac{1}{6}R)G(x, x')=\delta(x, x'). \tag{5}$$

We refer to Dowker and Critchley (1977) for notation.

Because of our choice of a conformal coupling, under the conformal transformation (2), the Green functions in the two metrics are related by

$$G(x, x')=\frac{a}{R(t)}\tilde{G}(\tilde{x}, \tilde{x}')\frac{a}{R(t')} \tag{6}$$

with $x=(t, \boldsymbol{x})$, $\tilde{x}=(\tau, \boldsymbol{x})$, and G and $\tilde{G}$ refer to the RW and Einstein metrics respectively.

The massless finite-temperature Green function for a scalar gas on an Einstein universe of radius a was given in Dowker and Critchley (1977) as an image sum of zero-temperature Green functions,

$$\tilde{G}_{\beta_0}(\tilde{x}, \tilde{x}')=\sum_{m=-\infty}^{\infty}\tilde{G}_{\infty}(\tilde{x}, \tilde{x}'-im\beta_0\lambda) \tag{7}$$

where

$$\tilde{G}_{\infty}(\tilde{x}, \tilde{x}')=\frac{-\mathrm{i}}{4\pi^2 a\sin(s/a)}\sum_{n=-\infty}^{\infty}\frac{s+2\pi na}{\sigma_n^2} \tag{8}$$

is the zero-temperature Green function, with $\sigma_n^2=(\tau-\tau')^2-(s+2\pi na)^2-\mathrm{i}\epsilon$, s the geodesic distance on S^3, λ the time-like unit vector (1, 0, 0, 0), and $\beta_0=(kT_0)^{-1}$. We would now propose to use (6) with (7) to define a finite-temperature Green function on the RW metric. Before doing this, however, we shall discuss some of the physics involved.

If we write any Green function as

$$\tilde{G}(\tilde{x}, \tilde{x}')=\mathrm{i}\frac{\langle\mathrm{out}|T\{\tilde{\phi}(\tilde{x})\tilde{\phi}(\tilde{x}')\}|\mathrm{in}\rangle}{\langle\mathrm{out}|\mathrm{in}\rangle} \tag{9}$$

with $|\mathrm{out}\rangle$ and $|\mathrm{in}\rangle$ any states, then, since under the conformal transformation (2) (see Parker 1973 for further details on conformal transformations)

$$\tilde{\phi}(\tilde{x})\rightarrow\phi(x)=\frac{a}{R(t)}\tilde{\phi}(\tilde{x}), \tag{10}$$

the relation (6) holds only if the $|\mathrm{out}\rangle$ and $|\mathrm{in}\rangle$ states appearing in $G(x, x')$ are identical to those of $\tilde{G}(\tilde{x}, \tilde{x}')$ (up to a normalisation factor). We could say that the $|\mathrm{out}\rangle$ and $|\mathrm{in}\rangle$ states used to define the Green functions are conformally invariant.

For the particular case of the finite-temperature Green function (7) we have

$$\tilde{G}_{\beta_0}(\tilde{x}, \tilde{x}')=\mathrm{i}\langle T\{\tilde{\phi}(\tilde{x})\tilde{\phi}(\tilde{x}')\}\rangle_{\beta_0} \tag{11}$$

where

$$\langle\hat{A}\rangle_{\beta_0}=\frac{\mathrm{Tr}(\mathrm{e}^{-\beta_0\hat{H}}\hat{A})}{\mathrm{Tr}(\mathrm{e}^{-\beta_0\hat{H}})}, \tag{12}$$

Tr denoting the usual Fock space trace and H the time-independent second-quantised Hamiltonian. Before using our proposed finite-temperature RW Green function we must therefore verify that the vacuum and many-particle states defined by a mode expansion of the scalar field in Einstein space correspond under the transformation (10) to vacuum and many-particle states, respectively, in the RW universe. This point has been discussed by Ford (1975) (see also Parker 1969). Because of our choice of conformal coupling, it turns out that the above one–one correspondence exists, resulting in time-independent creation and annihilation operators for the quanta in the RW metric. This in turn means that there is no particle creation induced by the expansion. This would not be true if the conformally invariant equation (4) were replaced by the minimal one; in this case particle creation would occur because the modes would not have simple transformation properties, and a recognised equilibrium configuration in Einstein space would not be recognisable as such in RW space.

We therefore *define* our finite-temperature Green function in a RW universe as the averaged sum of many-particle state matrix elements of the time ordered product, the averaging being the usual statistical averaging in the conformally related static Einstein universe with a time-independent Hamiltonian as in (12). That is

$$G_{\beta_0}(x, x') = \frac{a}{R(t)} \tilde{G}_{\beta_0}(\tilde{x}, \tilde{x}') \frac{a}{R(t')}. \tag{13}$$

3. Calculation of $\langle \hat{T}_{00} \rangle_{\beta_0}$

With our finite-temperature Green function defined as above we can proceed to calculate $\langle \hat{T}_{tt} \rangle_{\beta_0}$. For the special case of a RW metric we have

$$\langle \hat{T}_{tt} \rangle_{\beta_0} = -\frac{\mathrm{i}}{2} \lim_{x' \to x} \left[\nabla_t \nabla_{t'} - \nabla_t \nabla_t - \frac{\ddot{R}(t)}{R(t)} + \left(\frac{\dot{R}(t)}{R(t)} \right)^2 \right] G_{\beta_0}(x, x'), \tag{14}$$

the dots denoting differentiation with respect to t. As is obvious from (7) and (8), (14) can be rewritten as a double sum over m and n of certain coincidence limits. For the moment we shall drop the $m = 0$ contribution to (7) and (14) since the divergencies of the coincidence limit of (14) reside solely in this zero-temperature part. Later we will simply add on any remaining finite contribution to $\langle \hat{T}_{tt} \rangle_\infty$ after an appropriate renormalisation of this zero-temperature quantity.

The calculation of (14), excluding the $m = 0$ part, is straightforward and yields

$$\langle \hat{T}_{tt} \rangle_{\beta_0} - \langle \hat{T}_{tt} \rangle_\infty = \langle \hat{T}_{tt} \rangle_{\beta_0}^{\infty} + \langle \hat{T}_{tt} \rangle_{\beta_0}^{R}, \tag{15}$$

with

$$\langle \hat{T}_{tt} \rangle_{\beta_0}^{\infty} = \tfrac{1}{30} \pi^2 \beta^{-4} \tag{16a}$$

and

$$\langle \hat{T}_{tt} \rangle_{\beta_0}^{R} = \frac{2}{\pi^2} \beta^{-4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{(12n^2\xi^2 - m^2)}{(m^2 + 4n^2\xi^2)^3} + \frac{4m^2(m^2 - 20n^2\xi^2)}{(m^2 + 4n^2\xi^2)^4} \right) \tag{16b}$$

being the infinite space (Planckian) and correction (due to the compactness of S^3) terms respectively. The β which appears here is

$$\beta = \beta_0 \frac{R(t)}{a} = (kT)^{-1} \tag{17}$$

corresponding to the expected Tolman temperature arising from $g_{\tau\tau}$ in (2) and describing the cooling of the gas due to expansion. ξ is the dimensionless parameter

$$\xi = \pi a\beta_0^{-1} = \pi R(t)\beta^{-1}. \tag{18}$$

It is immediately recognised that (15) is identical to the energy density obtained for a gas at temperature T in a static Einstein universe of radius R (Dowker and Critchley 1977).

It is interesting to note that the correct term can be written alternatively as $V^{-1}(F'+T_0S')$ or $V^{-1}(F'+TS'')$ where $V=2\pi^2R^3(t)$ is the volume of S^3 and

$$F' = \frac{4}{\pi^4}\frac{\xi^4}{R(t)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(12n^2\xi^2-m^2)}{(m^2+4n^2\xi^2)^3} \equiv R^{-1}(t)f(\xi), \tag{19a}$$

$$S' = -\left(\frac{\mathrm{d}F'}{\mathrm{d}T_0}\right)_V = -k\pi\frac{a}{R(t)}\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}, \tag{19b}$$

and

$$S'' = -\left(\frac{\mathrm{d}F'}{\mathrm{d}T}\right)_V = -k\pi\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}. \tag{19c}$$

If we choose the identification (19*c*) then the total entropy is the same as that obtained in the Einstein case (Dowker and Critchley 1977). This corresponds to Ehlers' (1971) requirement of zero entropy production for an equilibrium distribution function.

$G_{\beta_0}(x, x')$ in (13) is not periodic in imaginary t nor in imaginary τ. It is, however, approximately periodic in imaginary τ with period β_0 if $\beta_0R(t)/a \ll 1$, the satisfaction of this condition also implying that $G_{\beta_0}(x, x')$ will be approximately periodic in imaginary t with period β. This approximate periodicity would then correspond to the usual properties of thermal Green functions in static manifolds (Gibbons and Perry 1977, Dowker 1977, Dowker and Kennedy 1978). We point out here that our condition for approximate periodicity is slightly different from that stated by Gibbons and Perry (1977).

The surprising aspect of the result (15), (16), is that, even though the manifold is non-static, the energy density appears to mimic that of an 'instantaneous' static Einstein universe (Dowker and Critchley 1977). Alternatively one can view this fact as a cancellation of the extra differential operators in (14), due to the non-static nature of the metric, against the non-periodic parts of $G_{\beta_0}(x, x')$. This 'non-static in static's clothing' result strengthens our arguments supporting the form (13).

The finite part of $\langle\hat{T}_{tt}\rangle_\infty$ is chosen to be that obtained by the point splitting of Bunch and Davies (1977) as

$$\langle\hat{T}_{tt}\rangle_\infty^{\mathrm{ren}} = (480\pi^2)^{-1}\left[\frac{\dot{R}\dddot{R}}{R^2} - \frac{1}{2}\left(\frac{\ddot{R}}{R}\right)^2 + \left(\frac{\dot{R}}{R}\right)^2\left(\frac{\ddot{R}}{R}\right) - \left(\frac{\dot{R}}{R}\right)^4 + R^{-4}\right]. \tag{20}$$

The last term in this square bracket is just the partial Casimir term first calculated by Ford (1975) and, as in the Einstein case (Dowker and Critchley 1977), for large ξ, corresponding to a high temperature T or a large radius R, it can be shown that the correction term (16*b*) tends to minus this partial Casimir density, while for small ξ it tends to minus the Planck density. $\langle\hat{T}_{tt}\rangle_{\beta_0}^{\mathrm{ren}}$ is then obtained by taking the sum of (15) and (20).

4. Discussion

We present the above as an example of how to define thermal equilibrium on non-static manifolds which are conformally static. The next step would be to use $\langle \hat{T}_{tt} \rangle_{\beta_0}^{\text{ren}}$ in Einstein's equations, and try to solve the back reaction problem self-consistently leading to $R(t)$ in terms of a, β_0, and various other initial data. If $\langle \hat{T}_{tt} \rangle_{\beta_0}^{\text{ren}}$ enjoyed the conformal transformation properties of its unrenormalised counterpart then this self-consistency would reduce to the scalar version of the case studied by Al'taie and Dowker (1978) for the Einstein universe, hence providing a relationship between the radius and (Tolman) temperature for the RW universe. The time dependence in $R(t)$ would not be exhibited by this means. The presence of $\langle \hat{T}_{tt} \rangle_{\infty}^{\text{ren}}$ alters this conformal self-consistency argument, but one would expect that at high temperatures its effect would be small leading to the same qualitative behaviour as that obtained by Al'taie and Dowker (1978).

After completion of this work the author's attention was directed to a paper by Cooke (1977) in which similar views are expressed concerning thermal equilibrium in a RW universe.

Acknowledgments

The author would like to thank Stuart Dowker for several discussions and for reading the proof, and the Science Research Council for providing support.

References

Al'taie M B and Dowker J S 1978 *Phys. Rev.* to be published
Bunch T S and Davies P C W 1977 *Proc. R. Soc.* A **356** 569
Cooke J H 1977 *Am. J. Phys.* **45**/12 1168
Dowker J S 1977 *J. Phys. A: Math. Gen.* **10** 115
Dowker J S and Critchley R 1977 *Phys. Rev.* D **15** 1484
Dowker J S and Kennedy G 1978 *J. Phys. A: Math. Gen.* **11** in the press
Ehlers J 1971 *General Relativity and Cosmology* ed R K Sachs (New York: Academic) p 1
Ford L H 1975 *Phys. Rev.* D **11** 3370
Gibbons G W and Perry M J 1977 *Proc. R. Soc.* in the press
Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Spacetime* (New York: Cambridge University Press)
Israel W 1972 *General Relativity* ed L O'Raiferotaigh (London: Oxford University Press)
Parker L 1969 *Phys. Rev.* **183** 1057
—— 1973 *Phys. Rev.* D **7** 976